

The *E*-Plane Taper Junction in Rectangular Waveguide

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Abstract—This paper compares formulas for reflection in rectangular waveguide from an *E*-plane junction with radial guide. There is a major discrepancy between results for the junction parameters from earlier results of the author and those in the *Waveguide Handbook*. The two sets of results are discussed; the Handbook result is apparently in error, but the formulas do not seem capable of reconciliation. Limitations and slight improvements on the author's earlier result are given, together with supporting measurements taken at *X* band.

I. INTRODUCTION

THE STRAIGHT GUIDE to taper section is a common element in microwave hardware. Its properties were examined by the author in the late 1940's, and results of a rather simple analysis were published in 1949 [1]. They were used to design a matched transition at the junction between a waveguide and horn, and measurements confirmed the theoretical predictions quite closely. In 1951, results of a much more sophisticated quasi-static analysis were published [2], and presumably both sets of results have been used by engineers in the intervening period. It was recently drawn to the author's attention that the two analyses are not in agreement, and attempts at explaining the differences are reported on here.

II. APPROXIMATE FIELD-MATCHING FORMULA

Since the earlier quoted publication may not be readily available now and the analysis is quite short, it is repeated here for convenience. Fig. 1 shows the guide-to-taper junction, with the vertex of the tapered section located a distance r_0 beyond the junction. In the rectangular guide it is assumed that only the dominant mode need be taken into account, so that the relevant fields are

$$\begin{aligned} E_y &= (e^{-jk'z} + Re^{jk'z}) \sin(\pi x/a) \\ H_x &= -\hat{\eta}(e^{-jk'z} - Re^{jk'z}) \sin(\pi x/a)(k'/k) \end{aligned} \quad (1)$$

where

a	guide width,
k	$2\pi/\lambda$,
k'	$2\pi/\lambda_g$,
λ, λ_g	free space and guide wavelengths,
R	dominant mode reflection coefficient,
$\hat{\eta} = (\epsilon_0/\mu_0)^{1/2}$	the wave admittance in free space.

A time dependence $e^{j\omega t}$ is suppressed.

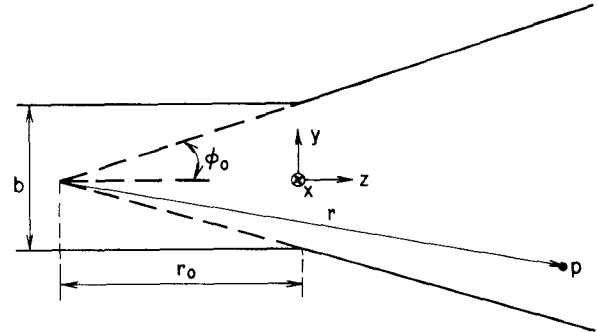


Fig. 1. Guide-to-taper junction.

In the tapered section, again only a dominant cylindrical mode is assumed, and it takes the form

$$\begin{aligned} E_\phi &= TH_1^{(2)}(k'r) \sin(\pi x/a) \\ H_x &= -\hat{\eta}TH_0^{(2)}(k'r) \sin(\pi x/a)(jk'/k) \end{aligned} \quad (2)$$

where T is a transmission coefficient.

The two field forms must be equated at the junction. The continuity of tangential fields has to be applied at all points of the junction cross section, and it is clear that this cannot be done exactly with the assumed fields. However, to the extent that r varies little over the junction, an approximate result can be obtained by equating the fields at the junction center $r=r_0$. The resulting equations are

$$1 + R = TH_1^{(2)}(k'r_0) \quad 1 - R = jTH_0^{(2)}(k'r_0). \quad (3)$$

Hence, the junction admittance Y_{in} (normalized with respect to the waveguide) is

$$Y_{in} = (1 - R)/(1 + R) = jH_0^{(2)}(k'r_0)/H_1^{(2)}(k'r_0). \quad (4)$$

Now, since it has been assumed that r varies little over the junction, we must consider $k'r_0$ to be large (just how large is considered later), so that the asymptotic forms of the Hankel functions can be used. Finally, using the relation $r_0 = \frac{1}{2}b \cot \phi_0$, where ϕ_0 is the half-angle of the taper, gives the result found in [1]

$$Y_{in} \approx 1 + j/(k'b \cot \phi_0). \quad (5)$$

III. DISCUSSION OF THE APPROXIMATE FORMULA

The parallel susceptance from (5) is capacitive, but since k' occurs in the denominator, the slope is opposite that of a genuine capacitive element. Hence, as discussed in the reference, the taper behaves more like a *negative*

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inductance shunted across the guide at the junction, and can be matched on a broad-band basis by a simple inductive diaphragm whose dimensions are readily calculated. Early experiments indicated that this broad-band cancellation was in fact achieved, and final results on a matched horn were measured with the taper junction matched in this way [3].

Clearly, a true negative inductance cannot exist, and the above result is partially due to the approximations used. The matter can be rectified very simply by inverting (5) to give to the same degree of approximation as used in going from (4) to (5):

$$Z_{in} = 1/Y_{in} \approx 1 - j/(k'b \cot \phi_0). \quad (6)$$

The interpretation, now, is simply that the junction behaves like an ordinary series capacitance. Fairly broad-band matching with a shunt inductance follows from the approximate constant-resistance network equivalence of the arrangement [3].

For the results to be valid, two conditions must be satisfied. For the asymptotic formulas of the Hankel functions to be used, we need

$$k'r_0 = \frac{1}{2}k'b \cot \phi_0 \gg 1. \quad (7)$$

For r to be approximately constant over the cross section, we need in addition

$$k'[(r_0^2 + b^2/4)^{1/2} - r_0] \ll 1 \quad (8)$$

which can also be written

$$k'b \tan (\phi_0/2) \ll 1. \quad (9)$$

For most rectangular waveguides, used near the center of their range, we have $a \approx 2b \approx \lambda_g/2$ so that $k'b$ in the above equations is around $\pi/2$. Thus both (7) and (9) in the usual guide range can be considered as setting limits on ϕ_0 . The limits set by (7) are easy to assess from tables of Bessel functions. Thus we find

$$\begin{aligned} H_1^{(2)}(x)/jH_0^{(2)}(x) &= 1.02 - j0.24, & \text{when } x = 2 \\ &= 1.07 - j0.45, & \text{when } x = 1 \end{aligned}$$

whereas, (6) would give $1 - j0.25$ and $1 - j0.5$, respectively, in the two cases. Clearly, if we can tolerate an error of a few percent, we can take a value $x = \pi/2$ to give

$$k'b \cot \phi_0 \geq \pi \quad (10)$$

as one limit.

The expression on the left of (9) is twice the phase deviation from uniformity over the aperture. If we denote the permissible phase error by $\Delta\psi$, and use the lower limit in (10), then (9) gives

$$\tan \phi_0 \tan (\phi_0/2) \leq 2\Delta\psi/\pi. \quad (11)$$

Thus $\phi_0 = 20^\circ$ corresponds to a phase error of $\Delta\psi = 5.7^\circ$, which seems a reasonably small error for a total taper angle of 40° . (Most of the tapers used in the experiments reported in [3] were well below this size.) If we use $k'b \approx \pi/2$ in (10) we would need $\tan \phi_0 \leq 1/2$, or $\phi_0 \leq 27^\circ$.

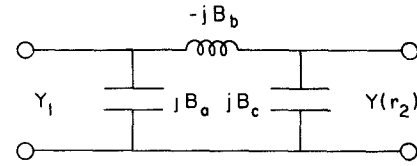


Fig. 2. Quasi-static equivalent circuit.

Thus for the usual values of waveguide dimensions, it is (9) that sets the more severe limit on ϕ_0 .

In summary, (6) for the junction impedance is to be preferred to (5) for the admittance; (7) and (9) set limits for b and ϕ_0 , such that, for most guides, an upper limit to ϕ_0 around 20° is encountered. But for *any* value of $k'b$ there will always be a value of ϕ_0 small enough to ensure that both restrictions are met, and as $\phi_0 \rightarrow 0$, the results should be quite accurate in any case. Results for comparison with the Handbook equations will, therefore, be taken for fixed b/λ_g and small taper angle, where the above formulas should be the most accurate.

It should, perhaps, be pointed out that (8) is taken from a phase requirement, and that there will also be a corresponding amplitude variation. This, however, will be quite small when (8) is satisfied, and it is the phase discrepancy which sets the limit to the accuracy of the field matching. Of course, the present method does not yield the increased field concentration adjacent to the convex corner at the junction, such as would be expected to exist from a consideration of the static case.

IV. THE WAVEGUIDE HANDBOOK FORMULA

The circuit parameters were stated to be obtained by a "simple equivalent static method," but no details were published. It leads to a more complicated equivalent circuit than the method of Section II; for convenience it is repeated here in Fig. 2, with the circuit parameters given by

$$Y(r_2)/Y_1 = \sin(\theta)/\theta \quad (12)$$

$$B_a/Y_1 = (2\pi b'/\lambda_g \theta) \log(\theta \operatorname{cosec} \theta) \quad (13)$$

$$B_c/Y_1 = (2b'/\lambda_g) [\gamma + \Psi(\theta/\pi)] \quad (14)$$

$$B_b/Y_1 = (\lambda_g/\pi b') \frac{\sin \theta}{\theta} \frac{\sin \theta}{1 - \sin(2\theta)/2\theta} \quad (15)$$

where γ is Euler's constant, $0.5772 \dots$.

The notation of the Handbook has been retained; θ is the same as ϕ_0 , and b' is the same as $b/2$ of the previous sections. (The Handbook notation is changed here to b' to prevent confusion with b in comparing results.) The Ψ function is not, as commonly used, the logarithmic derivative of the gamma function, but of the factorial function. Thus $\Psi(x) = d/dx \log \Gamma(1+x)$, and is the function tabulated in Jahnke and Emde's "Table of Functions" [4]. It differs from the usual logarithmic derivative by an extra term $1/x$ coming from $\Gamma(1+x) = x\Gamma(x)$. From the known properties of the gamma function [5] we have

$$\frac{d}{dx} \log \Gamma(1+x) = -\gamma + x \sum_1^{\infty} \frac{1}{n(n+x)} = -\gamma, \quad \text{at } x=0$$

$$\frac{d^2}{dx^2} \log \Gamma(1+x) = \sum_1^{\infty} \frac{1}{(x+n)^2} = \pi^2/6, \quad \text{at } x=0.$$

Hence, using a Taylor expansion for small x

$$\Psi(x) = -\gamma + x\pi^2/6 + \dots \quad (16)$$

Using these results we find, for small θ , that

$$Y(r_2)/Y_1 = 1 + O(\theta^2)$$

$$B_a = B_c = (2\pi b'\theta/6\lambda_g)[1 + O(\theta^2)] \quad (17)$$

$$B_b = (3\lambda_g/2\pi b'\theta)[1 + O(\theta^2)].$$

As $\theta \rightarrow 0$, B_a and B_c approach zero, B_b (which is a series element) becomes infinite, and the circuit reduces to a matched termination. But, for θ small but not zero, the equivalent circuit gives, for the input admittance:

$$Y_{in} = jB_a + [1/(1+jB_c) + 1/(-jB_b)]^{-1}$$

$$\approx 1 + j(B_a + B_c - 1/B_b) \quad (18)$$

where B_a , B_c , and $1/B_b$ are all small. For small θ , this is the result for comparison with (5). Unfortunately, when [17] is used in [18], we get a susceptance $(2\pi b'\theta/\lambda_g)(1/6 + 1/6 - 1/3) = 0$, so there is no surviving term which is first order in θ , and a proper comparison is not possible.

V. DISCUSSION OF THE HANDBOOK FORMULA

According to the *Waveguide Handbook* the formulas are valid for $2b'/\lambda_g \ll 1$, with an estimated accuracy of a few percent when $2b'/\lambda_g < 0.1$. There is no stated restriction on θ , and it should certainly be permissible to use the small θ limits.

In correspondence, Dr. Marcuvitz, who was kind enough to go through his early notes on the subject, writes, "I suspect the equivalent circuit parameter for B_b may be written down incorrectly, since a result in my notes for a single lumped susceptance approximation is $j(B_a + B_c)$, which apparently is not the result derivable from the equivalent circuit and appears to indicate that the B_b term in the Handbook is not correct."

The term $(B_a + B_c)$ alone is $2\pi b'\theta/3\lambda_g$, whereas, in the same notation, (5) gives $\theta\lambda_g/4\pi b'$. The former expression is supposedly valid for small θ and small (b'/λ_g) , and the latter for small θ and large $(b'/\theta\lambda_g)$. Hence, the results can be in no way reconciled in the common range of small b'/λ_g and small enough θ . The coefficients of θ are seen to be the same when $b(=2b')=0.39\lambda_g$, which is much larger than the usual guide size. For $b=\lambda_g/4$, the formulas differ by a factor 2.43, the Handbook result being the smaller. Both these values of b are greater than the $0.1\lambda_g$ value suggested in the Handbook for accuracies of a few percent. For even smaller values of b/λ_g , the discrepancy between the two results is, of course, still greater.

Some theoretical support for the structure of the Handbook formula can be obtained in a simple way from a consideration of the static limit ($\lambda_g \rightarrow \infty$) as follows. For

a given value of electric field, the voltage across the guide is proportional to b , whilst the excess electric charge in a region near the junction would be proportional to b also. Thus the excess capacitance C of the junction region is independent of b . The guide impedance Z_g is proportional to $(b\lambda/\lambda_g)$ so the parameter ωCZ_g , which measures the junction susceptance effect, would thus be proportional to (b/λ_g) , as found above, but only in the limit for large λ_g . The field-matching formula is contingent on $(b/\phi_0\lambda_g)$ being large, and is not, in this sense, a static formula. In fact it depends essentially on the wave in the tapered section being in a "propagating" condition at the throat of the taper, which is a dynamic, not a static, requirement. This may account for the basic difference in the two results, but does not indicate which should be the more accurate in the common region of moderate b/λ_g and small enough θ or ϕ_0 .

VI. MEASURED RESULTS

The measurements reported in [3] were taken some 30 years ago, and are no longer available. In order to help settle the validity of either of the two junction formulas, an X-band taper section was constructed with a taper semiangle ϕ_0 of 12° , well within the expected limits of validity for (6). The taper length was 12 in, as shown in Fig. 3, and the reflection into the guide was measured from 8.2–11.1 GHz. Because of the mismatch at the horn mouth, there is a beat between the horn mouth and the junction, the number of maxima corresponding to the 12-in taper length being 8 in the measurement band. The results are shown in Fig. 4.

The magnitude of the horn reflection is known from the well-known Wiener-Hopf analysis [6] for a parallel-plate waveguide, and it should be fairly close for the tapered horn with an aperture in the E -plane of some 6 in. By combining the figures at the maxima in Fig. 4, with the values of the minima on either side, values can be obtained for $R_1 \pm R_2$, where R_1 and R_2 are, respectively, the magnitudes of the reflection from the horn mouth and the junction. There is a built-in ambiguity as to which is which, but by comparison with the theoretical figure from the horn, given by [6]

$$R_1 = (2a/\lambda - 2a/\lambda_g)e^{-1/2k'a} \quad (19)$$

the ambiguity can be removed.

The results of this calculation are shown in Fig. 5, from which it can be seen that

- 1) the measured and theoretical values for the horn mouth are in close agreement, the measurements being a little on the low side;
- 2) the measured values for the junction are close to, but a little lower than, the values indicated by (6), and the slopes are comparable;
- 3) there is no agreement with the Handbook values, either in magnitude or slope (the Handbook figure was taken from $B_a + B_c = 2\pi b'/3\lambda_g$).

The discrepancies between measurements and theory are

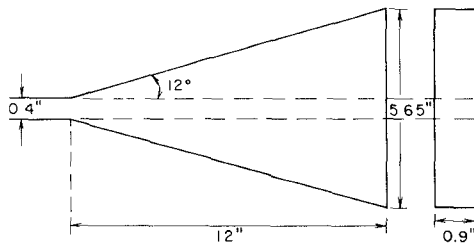


Fig. 3. Tapered section.

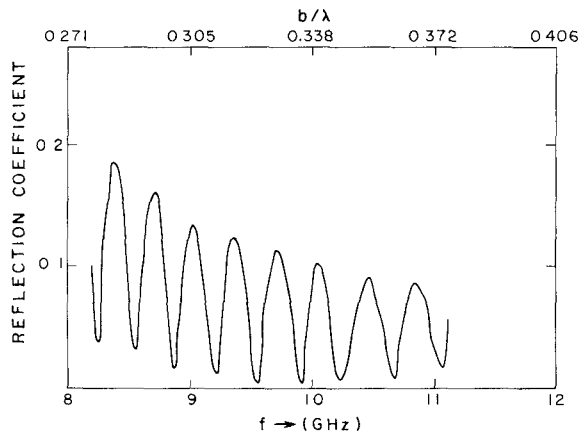


Fig. 4. Measured reflection from the taper and horn mouth.

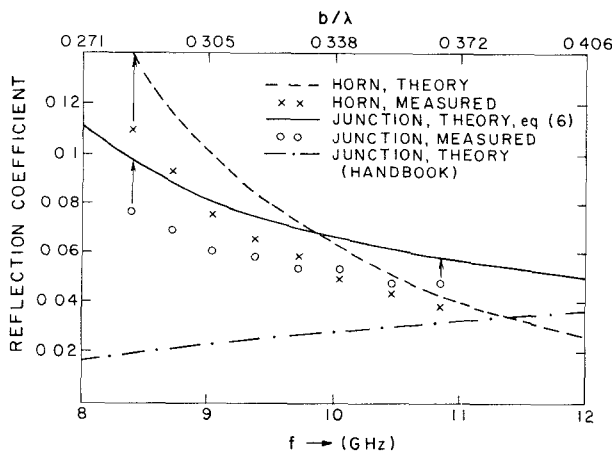


Fig. 5. Comparison of measured and theoretical values.

about 0.01–0.02 in reflection coefficient. It was estimated that the experimental error for the equipment used was about 0.01, but the consistency with which the measure-

ments fall below the theoretical values indicates either a small systematic error, or a slight overestimation by the theory. In any case, for the present purpose of distinguishing between the two available theories, the figures are quite adequate.

VII. CONCLUSIONS

It would appear that the Handbook equations are in error, at least as far as the series element is concerned, but even so, the other susceptance formulas appear to be incompatible with the measurements since the factor b/λ_g increases with frequency, whilst the measured values show a decrease. There seems to be no obvious way of reconciling these results. The measurements are, however, in close agreement with (5) and (6). The latter is preferred since the equivalent circuit reduces to a simple series capacitor, with ordinary (positive) capacitance. In the static limit, for a fixed taper angle, neither of these equations will ultimately apply. However, in the usual waveguide range and for taper angles that are not too great, they are well within their proper range of validity, and can be used with confidence for taper design.

ACKNOWLEDGMENT

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